Optimal Tuition and Investment Share in a Simple School–Student Model

Maksym Kutsenko

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Abstract

This note develops a simple model in which a school chooses tuition z and the share θ of tuition reinvested in education, while students decide whether to enroll based on their pre-school human capital. We derive the first-order conditions for the school's problem, characterize the equilibrium numerically, and illustrate how the optimal choices vary with the multiplication factor by which schooling increases human capital $(1 + \delta)$ for a given level of school investment and how much human capital is rewarded on the labor market(β) via an R implementation.

1 Introduction

Many educational institutions face the dual decision of setting tuition prices and allocating spending per student. Higher per-student spending improves graduate human capital but reduces net revenue. Students decide to enroll if their expected lifetime benefit exceeds tuition. We present a tractable framework capturing this multi-objective trade-off and provide numerical illustrations.

2 Model Setup

Model Overview

We begin by stating the two core optimization problems. First the school's objective function:

$$\max_{z,\theta} \; Y\Big(n(z),\, z\,\theta\Big) = n(z)\, z \;-\; \theta\, z\, n(z)\, \frac{a - HC^*}{a},$$

where a is a constant and HC^* denotes the pre-school human capital of the marginal student. This function very simply defines the profit a school makes as a function of the number of students attending the school times the tuition (Revenue) minus the cost which depends on the share of the tuition allocated to teaching, the number of students, and the human capital of entering students with the assumption that it is easier/faster to teach students with higher levels of human capital.

Now the student objective function:

$$\max_{d_i \in \{0,1\}} W_i = 2\beta\, HC_{0,i} + d_i\Big(\big[(1+\delta)(1+\theta) - 2\big]\beta\, HC_{0,i} - z\Big).$$

Here the student is choosing between not attending the school and earning $\beta\, HC_{0,i}$ for both period 0 and period 1, or attending the school, paying tuition z, and reaching a level of human capital of $(1+\delta)(1+\theta)HC_{0,i}$ which will get rewarded by a factor β.

These two expressions summarize the school's and students' decisions. Now let us we derive FOCs and characterize the equilibrium.

2.1 Student's Decision

Each prospective student has pre-school human capital $HC_{0,i} \in [0,1]$. Taking the FOC with respect to the schooling choice d_i in the students objective function it is clear that a student enrolls if

$$[(1+\delta)(1+\theta) - 2]\,\beta\, HC_{0,i} \geq z.$$

For the marginal student:

$$[(1+\delta)(1+\theta) - 2]\,\beta\, HC^* = z.$$

Define

$$C(\theta) = (1+\delta)(1+\theta) - 2, \quad HC^* = \frac{z}{\beta\, C(\theta)}, \quad n = 1 - HC^*.$$

2.2 School's Problem

The school sets tuition z and per-student investment share θ. Total enrollment is $n(z,\theta) = 1 - HC^*$. Revenue per student is z, and unit cost is

$$\theta\, z\, \frac{a - HC^*}{a},$$

with $a = 2$. Aggregate profit

$$Y(z,\theta) = n\big[z - \theta z \tfrac{2-HC^*}{2}\big] = z\Big(1 - \tfrac{z}{\beta C}\Big)\Big[1 - \theta + \theta\tfrac{z}{2\beta C}\Big].$$

3 Equilibrium Conditions

The interior first-order conditions

$$\frac{\partial Y}{\partial z} = 0, \quad \frac{\partial Y}{\partial \theta} = 0$$

yield two equations in (z,θ) that can be solved numerically. For example, with $\beta = 1$, $\delta = 2$, one finds

$$z^* \approx 1.42, \quad \theta^* \approx 0.52,$$

implying cutoff $HC^* \approx 0.55$ and enrollment $n \approx 0.45$.

4 R Implementation and Sensitivity to δ

We implement a grid search over δ and use `optim(..., method='L-BFGS-B')` to recover (z^*, θ^*) as functions of δ. Figure **??** shows the results.

```
library(plotly)

# 1) PARAMETERS & GRID
a          <- 2
delta_vals <- seq(0.1, 5, length.out = 1000)
beta_vals  <- seq(0.1, 5, length.out = 1000)

# 2) PROFIT FUNCTION
profit_fn <- function(z, theta, delta, beta, a) {
  C <- (1 + delta) * (1 + theta) - 2
  if (C <= 0) return(-1e8)
  HCstar <- z / (beta * C)
  if (HCstar < 0 || HCstar > 1) return(-1e8)
  n      <- 1 - HCstar
  n * (z - theta * z * (a - HCstar) / a)
}

# 3)  P R E ALLOCATE  MATRICES
z_mat     <- matrix(NA, nrow = length(beta_vals), ncol = length(
   delta_vals))
theta_mat <- matrix(NA, nrow = length(beta_vals), ncol = length(
   delta_vals))

# 4) OPTIMIZATION LOOP
for (i in seq_along(delta_vals)) {
  for (j in seq_along(beta_vals)) {
       <- delta_vals[i]
       <- beta_vals[j]
    obj <- function(par) {
      -profit_fn(par[1], par[2],   ,   , a)
    }
    fit <- optim(
      par    = c(1, 0.5),
      fn     = obj,
      method = "L-BFGS-B",
      lower  = c(1e-6, 1e-6),
      upper  = c(10,   0.999)
    )
    z_mat[j, i]     <- fit$par[1]
    theta_mat[j, i] <- fit$par[2]
  }
}
```

```r
# 5) INTERACTIVE SURFACE FOR z*
p1 <- plot_ly(
  x = ~delta_vals, y = ~beta_vals, z = ~z_mat,
  type = "surface"
) %>%
  layout(
    title = expression(z^"*"(delta, beta)),
    scene = list(
      xaxis = list(title = "delta"),
      yaxis = list(title = "beta"),
      zaxis = list(title = "z*")
    )
  )

# 6) INTERACTIVE SURFACE FOR theta*
p2 <- plot_ly(
  x = ~delta_vals, y = ~beta_vals, z = ~theta_mat,
  type = "surface"
) %>%
  layout(
    title = expression(theta^"*"(delta, beta)),
    scene = list(
      xaxis = list(title = "delta"),
      yaxis = list(title = "beta"),
      zaxis = list(title = "theta*")
    )
  )

# 7) Print widgets (in RStudio they ll pop up in the Viewer pane;
   in a browser they open in a new tab)

p1

p2
```

In Figure 1 we can see that there is a first plateau at otimal tuition = 1 for low values of δ and β. The platea happens roughly for $\delta * \beta \leq 1$. We also see a second plateau for extremely high values of β and δ at z = 10. This plateau happens at roughly $\delta * \beta \geq 16$. We can also see that there are certain areas with high δ low β where the tuition is actually below 1 (around 0.5). On the remainder of the plot, higher values of either β or δ increase the optimal tuition charged by schools.

When it comes to the optimal share of tuition devoted to students, Figure 2 shows a plateau at share = 0.5 for what seems to be roughly the same values of β and δ as in the previou plot (for $\delta * \beta \leq 1$). We can also see that the share decreases more steeply for what appears to be the values of δ and β such that $\delta * \beta \geq 16$. This suggests that while the tuition

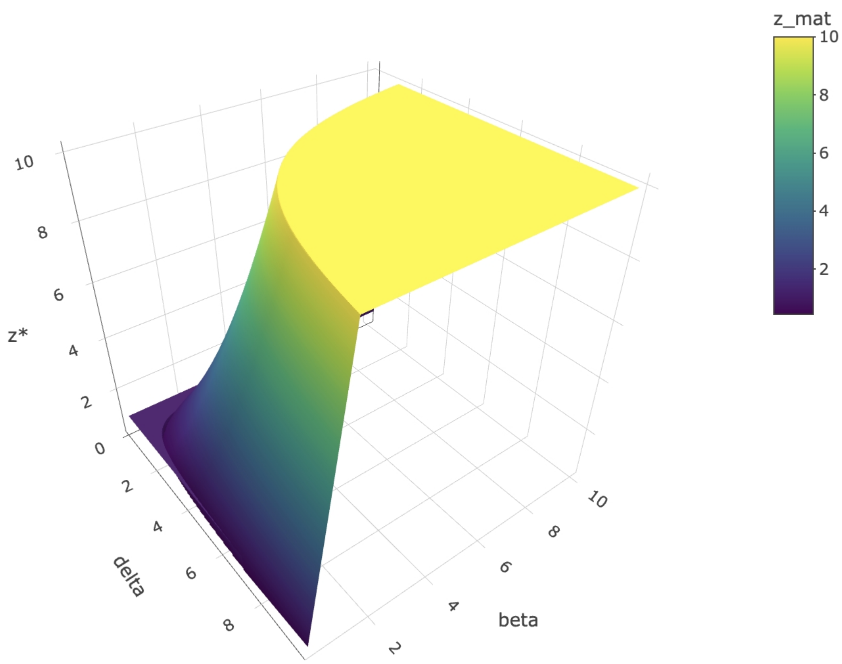

Figure 1: Optimal tuition

plateaus, schools decrease the share of the tuition allocated to students to maximize their profits. There is also this high δ low β area, but this time in this area the share allocated to students is lower. So in that area, we have low tuition low share allocated to students. One vert interesting area is the area where schools do not increase their students' productiivity by a lot (low δ) but human capital is highly rewarded on the job market. In this situation, it looks like the optimal behavior for schools is to devote a larger share of their resources to students.

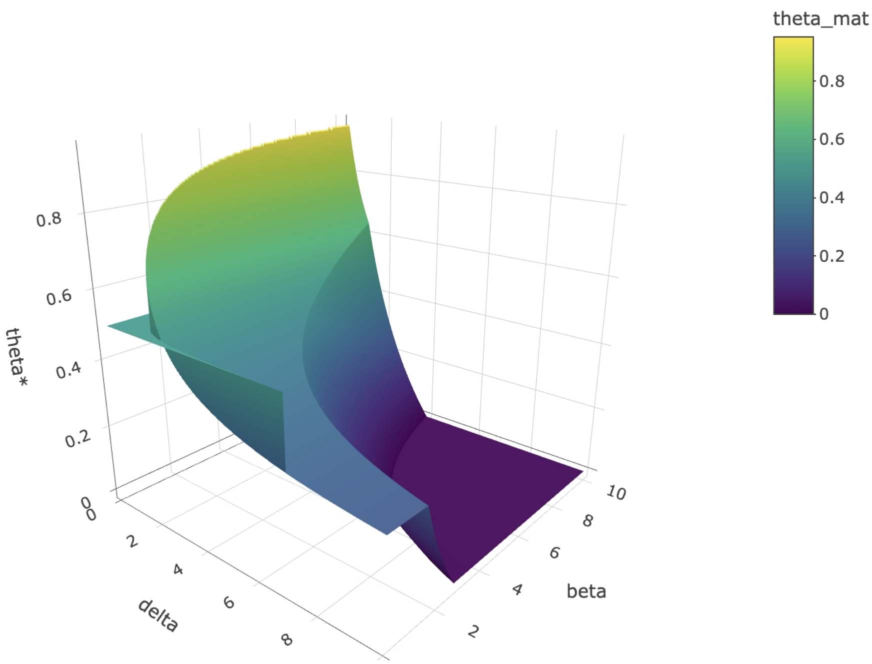

Figure 2: Optimal Share of Tuition devoted to Students

5 Conclusion

We have outlined a parsimonious model capturing tuition setting and reinvestment choices by a school facing profit and student enrollment trade-offs. We find 4 types of areas:

1. Low $\delta * \beta$: This area is characterized by low tuition fees and moderate shares allocated to students (50%).

2. High $\delta * \beta$: This area is characterized by very high tuition fees (10 times the ones in the low $\delta * \beta$ area) and very low shares allocated to students. This suggests that when the school has a very high impact on students' human capital accumulation, and human capital is very highly rewarded, schools can allow to allocate very little of their tuition to its students at the optimum.

3. The area in between these two is also interesting. As the productivity of schools increases, they increase their tuition but decrease the share allocated to students. Interestingly, it seems that for a given level of school productivity, higher levels of human capital reward on the labor market does not change the share of the tuition allocated to students.

4. Finally there are high school productivity (δ) low human capital reward areas, and low school productivity - high human capital reward areas. In these areas, as school productivity goes up and human capital reward goes down, both tuition and the share allocated to students goes down. Conversly, as schools get less productive, but human

capital gets increasingly rewarded on the labor market, tuition and shares allocated to students go up.